Extreme Covering Systems

CTNT 2022 Conference

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- A Cute Photo of My Cat

Introduction

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A covering system is called distinct if no two of the moduli are equal.

Motivation

Conjecture (de Polignac, 1849)

All odd integers ≥ 3 *can be written as* $2^k + p$ *for* $k \in \mathbb{N}$ *and* p *is either a prime or* 1

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Theorem (Erdős, 1950)

There exists an arithmetic progression consisting only of odd numbers, no term of which is of the form $2^k + p$*.*

The proof of the above is where Erdős invented covering systems.

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Proving or disproving this statement became *the minimum modulus problem*. For decades many mathematicians believed that indeed, it is possible to construct covering systems with arbitrarily large least modulus.

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Nielsen (2009) with 40 using a recursion

Owens (2014) with 42 (using over 10^{50} congruences)

Breakthrough

Turns out Erdős was wrong (gasp).

Theorem (Hough, 2015)

The minimum modulus in any distinct covering system does not exceed 10^{16}.

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(This number makes my work possible)

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If the minimum modulus of a distinct covering system is 2, then the largest modulus is at least 12.

Theorem (Krukenberg, 1971)

If the minimum modulus of a distinct covering system is 3, then the largest modulus is at least 36.

Main Results

Krukenberg said he proved the following but no proof has ever shown up in the literature:

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Theorem

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In the paper with Dr. Trifonov, we supply a proof.

Also, we proved the following:

Theorem (D. and Trifonov, 2022)

For each integer $m \geq 3$*, there is no distinct covering system with all moduli in the interval* $[m, 8m]$

Helpful Lemma

One of the tools we used:

Lemma

If $\mathcal{C}$ is the list of congruences in a covering system, then

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$$\sum_{n \in \mathcal{C}} \frac{1}{n} \geq 1.$$

The probability that a random integer is in a particular congruence class modulo n is exactly $\frac{1}{n}$, so if all of the probablilities do not add up to at least 1, then the list of congruences is not a covering.

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We find the remainders $r_1, r_2, \ldots, r_k$ when r is divided by $p_1^{a_1}, \ldots, p_k^{a_k}$ respectively.

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Then, $x \equiv r \pmod{n}$ is written $(d_1|\ d_2|\ \ldots|\ d_k)$ in our notation.

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Thus, for $x \equiv 6 \pmod{120}$ we have $r = 6$ and $n = 2^3 \cdot 3 \cdot 5$, and so

$$\begin{aligned} r &\equiv 6 \pmod{2^3} && \Rightarrow r_1 = 6 && \Rightarrow d_1 = 011_2, \\ r &\equiv 0 \pmod{3} && \Rightarrow r_2 = 0 && \Rightarrow d_2 = 0_3, \\ r &\equiv 1 \pmod{5} && \Rightarrow r_3 = 1 && \Rightarrow d_3 = 1_5. \end{aligned}$$

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So $x \equiv 6 \pmod{120}$ is written $(011|\ 0|\ 1)$.

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One more little note about the notation: if one or more of the exponents a_l in the factorizatizion $n = p_1^{a_1} \cdots p_k^{a_k}$ is zero, then we put $*$ in the lth position of the notation for the congruence.

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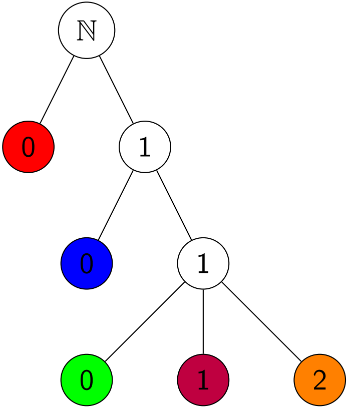
For example,

$$x \equiv 1 \pmod{10} \text{ is written } (1| \ * \mid 1).$$

Building a Distinct Covering Using a Tree



Normal notation → our notation

0 (mod 2) → (0)

1 (mod 4) → (10)

0 (mod 3) → $(*|\ 0)$

5 (mod 6) → $(1|\ 2)$

7 (mod 12) → $(11|\ 1)$

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Splitting the residue class $r \pmod{n}$ modulo p means that we replace it by p residue classes modulo np by consecutively appending the base-p digits $0, 1, \ldots, p-1$ in the position corresponding to p^{a+1} in the coordinate representation of the residue class.

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For example, if we split $(1|\ 1|\ 4)$ modulo 3, we obtain the 'fibers' $(1|\ 10, 11, 12|\ 4)$.

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Note, if you reduce a covering system modulo a prime, the resulting list of congruences will still be a covering (possibly not a disinct one).

So if you reduce a set of congruences that you think could be a covering modulo a prime, and end up with some integers left uncovered, then the original set of congruences cannot be a covering.

A Fun Example

Lemma

Let $\mathcal{C}$ be a covering system such that $p^a | L$, where L is the lcm of moduli, for some prime p and integer $a \geq 1$. Suppose that there are k congruences in $\mathcal{C}$ whose moduli are divisible by p^a. Then, if $k < p$, we can discard from $\mathcal{C}$ all congruences whose moduli are divisible by p^a, and will still have a covering.

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Let $\mathcal{C}$ be a distinct covering system with all moduli in the interval $[c, d]$. If p is a prime and a is a positive integer such that $p^a(p+1) > d$, then we can discard all congruences whose moduli are multiples of p^a and still have a covering.

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Using the lemma from the previous slide, we get that if there exists a distinct covering system with all of the moduli in the interval $[3, 35]$, then the set of moduli must be a subset of

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Lemma

Let $\mathcal{C}$ be a distinct covering system with all moduli in the interval $[c, d]$. If p is a prime and a is a positive integer such that $p^a(p+1) > d$, then we can discard all congruences whose moduli are multiples of p^a and still have a covering.

Using the lemma from the previous slide, we get that if there exists a distinct covering system with all of the moduli in the interval $[3, 35]$, then the set of moduli must be a subset of

$$\{3, 2^2, 5, 2 \cdot 3, 2^3, 2 \cdot 5, 2^2 \cdot 3, 3 \cdot 5, 2^2 \cdot 5, 2^3 \cdot 3, 2 \cdot 3 \cdot 5\}$$

From there you have to break it down into cases, which seems a bit tedious for this talk, so we'll move on.

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However, this last Lemma is one of the main ingredients for proving our theorem about the nonexistence of distinct covering systems with all of the moduli in the interval $[m, 8m]$ for $m \geq 3$.

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However, this last Lemma is one of the main ingredients for proving our theorem about the nonexistence of distinct covering systems with all of the moduli in the interval $[m, 8m]$ for $m \geq 3$.

Let's look at some of those details now, by combining the previous lemma with our old friend

Lemma

If $\mathcal{C}$ is the list of congruences in a covering system, then

$$\sum_{n \in \mathcal{C}} \frac{1}{n} \geq 1.$$

Assume that for some integer $m \geq 3$ there is a distinct covering $\mathcal{C}$ with all moduli in the interval $[m, 8m]$.

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Let L be the least common multiple of the moduli of the congruences in $\mathcal{C}_m$. By one of the lemmas, if $p^a | L$ for some prime p and a positive integer a, then the interval $[m, 8m]$ contains at least p multiples of p^a that are not multiple of p^{a+1}.

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Let L be the least common multiple of the moduli of the congruences in $\mathcal{C}_m$. By one of the lemmas, if $p^a | L$ for some prime p and a positive integer a, then the interval $[m, 8m]$ contains at least p multiples of p^a that are not multiple of p^{a+1}.

Since one of every p consecutive multiples of p^a are divisible by p^{a+1}, we get that the interval $[m, 8m]$ must contain at least $p + 1$ multiples of p^a.

Denote by $\mathcal{M} \subseteq [m, 8m]$ the set of moduli from the congruences in $\mathcal{C}_m$.

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$$n_p := \left\lfloor \frac{8m}{p} \right\rfloor - \left\lfloor \frac{m-1}{p} \right\rfloor = \frac{7m+1}{p} - \left\{ \frac{8m}{p} \right\} + \left\{ \frac{m-1}{p} \right\},$$

where $\{x\}$ denotes the fractional part of x.

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where $\{x\}$ denotes the fractional part of x.

Since $0 \leq \{x\} < 1$, if we assume $p \geq \sqrt{7m+1}$, we get

$$n_p < \frac{7m+1}{p} + 1 \leq \sqrt{7m+1} + 1 \leq p + 1.$$

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Therefore, if n is a modulus of one of the congruences in $\mathcal{C}_m$ (that is $n \in \mathcal{M}$), then all the prime divisors of n are less than $\sqrt{7m+1}$.

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Therefore, if n is a modulus of one of the congruences in $\mathcal{C}_m$ (that is $n \in \mathcal{M}$), then all the prime divisors of n are less than $\sqrt{7m+1}$.

Since, the density of integers covered by a congruence modulo n is $1/n$, and $\mathcal{C}_m$ is a covering, we get

$$\sum_{\substack{m \leq n \leq 8m, \\ P(n) < \sqrt{7m+1}}} \frac{1}{n} \geq \sum_{n \in \mathcal{M}} \frac{1}{n} \geq 1,$$

where $P(n)$ denotes the largest prime divisor of n.

Let

$$S_m = \sum_{n \in \mathcal{M}} \frac{1}{n} \quad \text{and} \quad T_m = \sum_{\substack{m \le n \le 8m, \\ P(n) < \sqrt{7m+1}}} \frac{1}{n}.$$

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We didn't need to check the sum for all values of m because we could make jumps by defining:

$$a_n = \begin{cases} \frac{1}{n}, & \text{if } P(n) < \sqrt{7n+1} \\ 0 & \text{otherwise.} \end{cases},$$

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Using this shortcut, the next value of T_m that we needed to calculate after T_{606000} was T_{286067}.

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Using this shortcut, the next value of T_m that we needed to calculate after T_{606000} was T_{286067}.

There were a few counterexamples for $m \in [3, 25]$, where $T_m > 1$, but these were fixed by considering the squares of some of the primes.

Open Problems and Further Work

Conjecture

If the least modulus of a distinct covering system is 5*, then its largest modulus is at least* 108*.*

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We can show that if the least modulus of a distinct covering system is 5, then its largest modulus is at least 84. However, the result is too weak, and the proof too long, to be included in our paper.

Open Problems and Further Work

Conjecture

If the least modulus of a distinct covering system is 5, *then its largest modulus is at least* 108.

We can show that if the least modulus of a distinct covering system is 5, then its largest modulus is at least 84. However, the result is too weak, and the proof too long, to be included in our paper.

I was able to use this weaker result to show the nonexistence of distinct covering systems with all of the moduli in the interval $[m, 9m]$ for $m \geq 3$ except for the numbers $m = 24$ and $m = 48$.

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